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APPROXIMATE THEORY OF VIBRATION OF CRYSTAL PLATES AT HIGH FREQUENCIES

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Abstract—The series expansion of displacement in terms of simple thickness modes is used to obtain approximate two-dimensional equations of motion for crystal plates from the three-dimensional theory of elasticity. Approximate theories from the first to the fourth order are presented. Dispersion curves for AT–cut quartz plate are explored and compared with the solution of the three-dimensional equations for an infinite plate.

1. INTRODUCTION

The formulation of approximate, two-dimensional equations of motion is based on the series-expansion method. Mindlin[1], using a power series expansion of displacement, obtained a set of equations which accommodates the frequencies of the first five modes of vibration. The increasingly complex mathematical forms in the formulation of higher order approximations, when a power series expansion is used, led to an expansion of displacement in a series of simple thickness modes for infinite plates[2]. The approximations of successively higher orders are then obtained without any complication, because the simple thickness modes are orthogonal.

This procedure was applied in the analysis of the vibration of isotropic plates[3] and is extended in this paper to the vibration of crystal plates. The dispersion curves for real, imaginary and complex wave numbers are explored in detail for AT–cut quartz plate and are compared with those obtained from Ekstein's[4] solution of the three-dimensional equations of elasticity for an infinite plate. The comparison reveals the close agreement between the respective sets of dispersion curves and indicates that the solutions of approximate equations, for bounded plates, will give reliable results over the same applicable range of frequencies.

2. THREE-DIMENSIONAL EQUATIONS

The variational form of the equations of motion, from which the two-dimensional equations are to be deduced, is

$$\int_V (T_{ij,i} - \rho u_{j,tt})\, \delta u_j \, \mathrm{d}V = 0 \tag{1}$$

where T_{ij}, u_j are the components of stress and displacement, respectively, and ρ is the mass density. In (1) the integration is over the region V for independent variations of u_j.

The constitutive equations are

$$T_{ij} = c_{ijkl} S_{kl} \tag{2}$$

where the c_{ijkl} are the elastic constants, S_{ij} are the components of strain and are expressed in terms of the u_j by

$$S_{ij} = \tfrac{1}{2}(u_{j,i} + u_{i,j}). \tag{3}$$

3. SERIES OF TWO-DIMENSIONAL EQUATIONS

The faces of an infinite plate are taken at $x_2 = \pm b$, the middle plane in the x_1, x_3 plane of an x_j rectangular coordinate system. The displacement components $u_j(x_i, t)$ are expanded in an infinite series of the simple thickness modes† as

$$u_j = \sum_{n=0}^{\infty} u_j^{(n)} \cos \frac{n\pi}{2}(1 - \eta) \tag{4}$$

where $\eta = x_2/b$ and the nth order displacements $u_j^{(n)}$ are functions of x_1, x_s and time t only. Then, from (1) and (4), using the identities

$$\begin{aligned} &\int_{-1}^{1} \sin \frac{n\pi}{2}(1 - \eta) \sin \frac{n\pi}{2}(1 - \eta)\, \mathrm{d}\eta = \delta_{nm} \\ &\int_{-1}^{1} \cos \frac{n\pi}{2}(1 - \eta) \cos \frac{n\pi}{2}(1 - \eta)\, \mathrm{d}\eta = \delta_{nm} \end{aligned} \tag{5}$$

and integrating over the interval $(-1, 1)$ with respect to η, one obtains

$$\int_A \sum_{n=0}^{\infty} \left(T_{ij,i}^{(n)} - \frac{n\pi}{2b} \bar{T}_{2j}^{(n)} + \frac{1}{b} F_j^{(n)} - \rho u_{j,tt}^{(n)} \right) \delta u_j^{(n)}\, \mathrm{d}A = 0 \tag{6}$$

where A is an area of a plate and

$$\begin{aligned} T_{ij}^{(n)} &\equiv \int_{-1}^{1} T_{ij} \cos \frac{n\pi}{2}(1 - \eta)\, \mathrm{d}\eta \\ \bar{T}_{ij}^{(n)} &\equiv \int_{-1}^{1} T_{ij} \sin \frac{n\pi}{2}(1 - \eta)\, \mathrm{d}\eta \\ F_j^{(n)} &\equiv \left[T_{2j} \cos \frac{n\pi}{2}(1 - \eta) \right]_{-1}^{1} = T_{2j}(b) - (-1)^n T_{2j}(-b) \end{aligned} \tag{7}$$

are defined as nth order components of stress with $\cos(n\pi/2)(1 - \eta)$ and $\sin(n\pi/2)(1 - \eta)$ as weighting functions, and nth order components of the face traction, respectively. Since (4) must be satisfied for every area A and arbitrary $\delta u_j^{(n)}$ then the nth order stress equations of motion are obtained from (4) as

$$T_{ij,i}^{(n)} - \frac{n\pi}{2b} \bar{T}_{2j}^{(n)} + \frac{1}{b} F_j^{(n)} = \rho u_{j,tt}^{(n)}. \tag{8}$$

† Series of vibrational modes which are independent of x_1, x_3 coordinates and correspond to traction free faces (see Mindlin[1]).

Defining

$$S_{ij}{}^{(n)} \equiv \tfrac{1}{2}(u^{(n)}_{i,j} + u^{(n)}_{j,i})$$
$$\bar{S}_{ij}{}^{(n)} \equiv \frac{n\pi}{4b}(\delta_{2i}u_j{}^{(n)} + \delta_{2j}u_i{}^{(n)}) \tag{9}$$

as nth order components of strain, one obtains from (3) and (4) components of strain

$$S_{ij} = \sum_{n=0}^{\infty}\left[S_{ij}{}^{(n)}\cos\frac{n\pi}{2}(1-\eta) + \bar{S}_{ij}{}^{(n)}\sin\frac{n\pi}{2}(1-\eta)\right]. \tag{10}$$

Substitution of (10) into (2), then, in turn, into (7) yields the nth order stress–strain relations

$$T_j{}^{(n)} = c_{ijkl}\left(S_{kl}{}^{(n)} + \sum_{m=0}^{\infty}A_{mn}\bar{S}_{kl}{}^{(m)}\right)$$
$$\bar{T}_j{}^{(n)} = c_{ijkl}\left(\bar{S}_{kl}{}^{(n)} + \sum_{m=0}^{\infty}A_{nm}S_{kl}{}^{(m)}\right), \tag{11}$$

where

$$\int_{-1}^{1}\sin\frac{m\pi}{2}(1-\eta)\cos\frac{n\pi}{2}(1-\eta)\,\mathrm{d}\eta = A_{mn} = \begin{cases} 0 &, \quad m+n \text{ even} \\ \dfrac{4m}{(m^2-n^2)\pi}, & \quad m+n \text{ odd.}\end{cases} \tag{12}$$

Two sets of stresses and strains are introduced in this theory. The $\bar{T}_{ij}{}^{(n)}$ components of stress provide the linkage between nth and $(n-1)$th order of equations while strain components $\bar{S}_{ij}{}^{(n)}$ are supplemental,[3] to $S_{ij}{}^{(n)}$.

The plate-strain energy density, which is defined as

$$\bar{U} \equiv \frac{1}{2}\int_{-1}^{1}T_{ij}S_{ij}\,\mathrm{d}\eta = \frac{1}{2}\int_{-1}^{1}c_{ijkl}S_{ij}S_{kl}\,\mathrm{d}\eta \tag{13}$$

becomes

$$\bar{U} = \frac{1}{2}\sum_{n=0}^{\infty}(T_{ij}{}^{(n)}S_{ij}{}^{(n)} + \bar{T}_{ij}{}^{(n)}\bar{S}_{ij}{}^{(n)}) \tag{14}$$

or

$$\bar{U} = \frac{1}{2}\sum_{n=0}^{\infty}c_{ijkl}\left[S_{ij}{}^{(n)}S_{kl}{}^{(n)} + \bar{S}_{ij}{}^{(n)}\bar{S}_{kl}{}^{(n)} + \sum_{m=0}^{\infty}(A_{mn}S_{ij}{}^{(n)}\bar{S}_{kl}{}^{(m)} + A_{nm}\bar{S}_{ij}{}^{(n)}S_{kl}{}^{(m)})\right]. \tag{15}$$

It can be noted that

$$T_{ij}{}^{(n)} = \frac{\partial\bar{U}}{\partial S_{ij}{}^{(n)}} \qquad \bar{T}_{ij}{}^{(n)} = \frac{\partial\bar{U}}{\partial\bar{S}_{ij}{}^{(n)}}. \tag{16}$$

Similarly, the plate-kinetic energy density becomes

$$\bar{K} \equiv \frac{1}{2}\int_{-1}^{1}\rho u_{j,t}u_{j,t}\,\sigma\eta = \tfrac{1}{2}\rho\sum_{n=0}^{\infty}u^{(n)}_{j,t}u^{(n)}_{j,t}. \tag{17}$$

The uniqueness of the solution of the approximate equations is established[2, 3] in a way similar to that of Neumann[5]. The sufficient conditions, in the absence of discontinuities and singularities, are listed in the previous paper[3], where also a theorem which shows the orthogonality of two functions $u_j^{(n)a}$ and $u_j^{(n)b}$ is presented.

4. TRUNCATION PROCESS AND ADJUSTMENTS

The approximate theories of various orders must be extracted from the infinite set of two-dimensional plate equations. This is achieved by using the following truncation process when for Nth order theory, with N any positive integer, is set

$$\left.\begin{array}{l} U_j^{(n)} = 0 \\ T_{ij}^{(n)} = \bar{T}_{ij}^{(n)} = 0 \end{array}\right\} \quad \text{for } n > N. \tag{18}$$

From the approximate theory of order N one obtains the dispersion curves for the propagation of straight crested waves in an infinite crystal plate with monoclinic symmetry, such as AT–cut quartz plate. By comparing the dispersion curves with those from the three-dimensional theory of elasticity[2, 4], it is found that the approximate theory always yields the "exact" cut-off frequencies since the simple thickness modes used in the series expansion of the displacement are the exact limits of the three-dimensional theory as wave lengths approach infinity. Two sets of dispersion curves match quite well for frequencies up to $\Omega = N + \frac{1}{2}$ and wave numbers $|z| = N + 1$, where the dimensionless frequency and wave number are defined by

$$\Omega = \omega \Big/ \left[\frac{\pi}{2b}\left(\frac{c_{66}}{\rho}\right)^{1/2}\right] \qquad z = \xi \Big/ \left(\frac{\pi}{2b}\right), \tag{19}$$

ω is a frequency, ξ is a wave number, except for the lowest flexural and the lowest extensional branches. Two correction factors, k_1 and k_2, are introduced in the strain and kinetic energy densities in order to achieve the better match of these branches:

$$\begin{aligned} 2\bar{U} = c_{ijkl}[&S_{ij}^{(0)}S_{kl}^{(0)} + \bar{S}_{ij}^{(0)}\bar{S}_{kl}^{(0)} + k_2{}^{p}S_{ij}^{(1)}S_{kl}^{(1)} + \bar{S}_{ij}^{(1)}\bar{S}_{kl}^{(1)} + S_{ij}^{(2)}S_{kl}^{(2)} \\ &+ \bar{S}_{ij}^{(2)}\bar{S}_{kl}^{(2)} + \cdots + S_{ij}^{(0)}(A_{10}k_1\bar{S}_{kl}^{(1)} + A_{30}\bar{S}_{kl}^{(3)} + \cdots) \\ &+ S_{ij}^{(1)}(A_{21}\bar{S}_{kl}^{(2)} + A_{41}\bar{S}_{kl}^{(4)} + \cdots) + S_{ij}^{(2)}(A_{12}\bar{S}_{kl}^{(1)} + A_{32}\bar{S}_{kl}^{(3)} + \cdots) \\ &+ S_{ij}^{(3)}(A_{23}\bar{S}_{kl}^{(2)} + A_{43}\bar{S}_{kl}^{(4)} + \cdots) + \cdots \\ &+ \bar{S}_{ij}^{(1)}(A_{10}k_1S_{kl}^{(0)} + A_{12}S_{kl}^{(2)} + A_{14}S_{kl}^{(4)} + \cdots) \\ &+ \bar{S}_{ij}^{(2)}(A_{21}S_{kl}^{(1)} + A_{23}S_{kl}^{(3)} + \cdots) + \bar{S}_{ij}^{(3)}(A_{30}S_{kl}^{(0)} + A_{32}S_{kl}^{(2)} + \cdots) + \cdots] \end{aligned} \tag{20}$$

and

$$2\bar{K} = \rho[k_2{}^{-p}u_{i,t}^{(0)}u_{i,t}^{(0)} + u_{i,t}^{(1)}u_{i,t}^{(1)} + u_{i,t}^{(2)}u_{i,t}^{(2)} + \cdots] \tag{21}$$

where

$$p = \cos^2\left(\frac{i\pi}{2}\right).$$

It is observed that k_1 is introduced into $\bar{U}$ for the strains $S_{ij}^{(0)}$ and $S_{ij}^{(1)}$, which are associated with the coefficients A_{10}, while k_2 is introduced into $\bar{U}$ for the term $S_{2j}^{(1)}S_{2j}^{(1)}$ and into $\bar{K}$ for the term $u_{2,t}^{(0)}u_{2,t}^{(0)}$.

The correction factor k_1 is determined from the comparison of the lowest flexural branch and the lowest extensional branch from the first order theory with those from the exact three-dimensional theory. In order to make the slopes of these branches to coincide when both the frequency and wave number approach zero, the value of k_1 must be set to

$$k_1 = \pi/4. \tag{22}$$

In order to make the phase velocities of the lowest flexural and the lowest extensional branches approach that of the surface wave propagating along the diagonal axis in a crystal as values of both the fequency and the wave number become large, k_2 must be equal, in the dimensionless form, to

$$k_2 = c^2\rho/c_{66} \tag{23}$$

where ρ is the density of a material and c is the velocity of the surface wave[6].

The adjusted energy densities $\bar{U}$ and $\bar{K}$ are positive and definite if, in addition to the usual requirements, $k_2 > 0$. The strain-displacement relations are not changed by correction coefficients. The stress equations of motion derived from $\bar{U}$ and $\bar{K}$ also remain unchanged except that the inertia term $\rho u^{(0)}_{2,tt}$ is replaced by $(\rho/k_2)u^{(0)}_{2,tt}$. The components of stress are derived from the adjusted $\bar{U}$ using (16) and include then the correction coefficients.

5. PLATE THEORIES OF SUCCESSIVELY HIGHER ORDERS

The plate theories from the first to the fourth order are extracted from the Nth order theory by using the truncation procedure and correction factors in $\bar{U}$ and $\bar{K}$.

The displacement equations of motion are further examined for the monoclinic crystal plates, such as AT–cut of quartz, where we set:†

$$c_{15} = c_{25} = c_{35} = c_{45} = c_{16} = c_{26} = c_{36} = c_{46} = 0. \tag{24}$$

The values of the other thirteen constants for the AT–cut quartz plate with x_1 the digonal axis in the plane of the plate and x_2 the axis normal to the plate are computed from Bechman's[7] six principal values and are, in units of 10^9 N/m^2,

$$\begin{array}{llll} c_{11} = 86{\cdot}74 & c_{44} = 38{\cdot}61 & c_{12} = -8{\cdot}26 & c_{23} = -7{\cdot}42 \\ c_{22} = 129{\cdot}77 & c_{55} = 68{\cdot}81 & c_{13} = 27{\cdot}15 & c_{24} = 5{\cdot}70 \\ c_{33} = 102{\cdot}84 & c_{66} = 29{\cdot}01 & c_{14} = -3{\cdot}65 & c_{34} = 9{\cdot}92 \\ & & & c_{56} = 2{\cdot}53. \end{array} \tag{25}$$

Two sets of equations, for the essentially symmetric and the essentially antisymmetric families of modes‡, respectively, are separated for the case of straight crested waves propagating in x_1 direction only for each order approximation. The dispersion curves for real, imaginary and complex numbers are explored in detail and are compared with those obtained from the three-dimensional theory[2]. This comparison is presented for the essentially antisymmetric family of modes in odd order theories (N=odd) and for the essentially symmetric family of modes in the even order theories (N=even).

The zero order theory is not a very suitable one due to the missing coupling between extensional and flexural modes[2] and therefore is not presented here.

† An abbreviated indicial notation is used. A pair of indices i, j ranging over integers 1–3 is replaced by one index p ranging over indices 1–6.

‡ The notation of [8] is employed.

First order theory

By setting $N = 1$ in (18) only the stress, strain and displacement components of the zero and the first orders are retained. Then the energy densities become

$$\begin{aligned} \bar{U}^{(1)} &= \tfrac{1}{2}[T_{ij}{}^{(0)}S_{ij}{}^{(0)} + T_{ij}{}^{(1)}S_{ij}{}^{(1)} + \bar{T}_{ij}{}^{(0)}\bar{S}_{ij}{}^{(0)}] \\ \bar{K}^{(1)} &= \tfrac{1}{2}\rho[k_2{}^{-p}u_{i,t}^{(0)}u_{i,t}^{(0)} + u_{i,t}^{(1)}u_{i,t}^{(1)}] \end{aligned} \tag{26}$$

and the stress equations of motion are:

$$\begin{aligned} T_{ij,i}^{(0)} + \frac{1}{b}F_j^{(0)} &= \rho k_2{}^{-p}u_{j,tt}^{(0)} \\ T_{ij,i}^{(1)} - \frac{\pi}{2b}\bar{T}_{2j}{}^{(1)} + \frac{1}{b}F_j{}^{(1)} &= \rho u_{j,tt}^{(1)}. \end{aligned} \tag{27}$$

The stress–displacement relations are readily obtained from (9) and (16) using the adjusted $\bar{U}$ by discarding those components $u_j{}^{(n)}$ for $n > 1$. The displacement equations of motion of the first order theory are then obtained by inserting the stress–displacement relations into (27):

$$\begin{aligned} &\tfrac{1}{2}c_{ijkl}\left[u_{k,l}^{(0)} + u_{l,k}^{(0)} + \frac{2}{b}k_1(\delta_{2l}u_k{}^{(1)} + \delta_{2k}u_l{}^{(1)})\right]_{,i} + \frac{1}{b}F_j{}^{(0)} = \rho k_2{}^{-p}u_{j,tt}^{(0)} \\ &\tfrac{1}{2}c_{ijkl}k_2{}^{p}(u_{k,l}^{(1)} + u_{l,k}^{(1)})_{,i} - \frac{\pi}{4b}c_{2ikl}\left[\frac{\pi}{2b}(\delta_{2l}u_k{}^{(1)} + \delta_{2k}u_l{}^{(1)}) + \frac{4}{\pi}k_1(u_{k,l}^{(0)} + u_{l,k}^{(0)})\right] \\ &\qquad + \frac{1}{b}F_j{}^{(1)} = \rho u_{j,tt}^{(1)}. \end{aligned} \tag{28}$$

Considering then the straight crested waves propagating in the x_1 direction we set:

$$\begin{aligned} u_j{}^{(0)} &= A_j{}^{(0)}e^{i(\xi x_1 - \omega t)} \\ u_j{}^{(1)} &= -iA_j{}^{(1)}e^{i(\xi x_1 - \omega t)} \\ F_j{}^{(0)} &= F_j{}^{(1)} = 0. \end{aligned} \tag{29}$$

Then the dispersion relations for monoclinic crystal plates are, in dimensional form,

$$\begin{vmatrix} \bar{c}_{11}z^2 - \Omega^2 & -\bar{c}_{12}\frac{4}{\pi}k_1 z & -\bar{c}_{14}\frac{4}{\pi}k_1 z \\ -\bar{c}_{12}\frac{\pi}{4}k_1 z & k_2 z^2 + \bar{c}_{22} - \Omega^2 & \bar{c}_{56}z^2 + \bar{c}_{24} \\ -\bar{c}_{14}\frac{4}{\pi}k_1 z & \bar{c}_{56}z^2 + \bar{c}_{24} & \bar{c}_{55}z^2 + \bar{c}_{44} - \Omega^2 \end{vmatrix} = 0 \tag{30}$$

and

$$\begin{vmatrix} z^2 - \frac{1}{k_2}\Omega^2 & \bar{c}_{56}z^2 & -\frac{4}{\pi}k_1 z \\ \bar{c}_{56}z^2 & \bar{c}_{55}z^2 - \Omega^2 & -\bar{c}_{56}\frac{4}{\pi}k_1 z \\ -\frac{4}{\pi}k_1 z & -\bar{c}_{56}\frac{4}{\pi}k_1 z & \bar{c}_{11}z^2 + 1 - \Omega^2 \end{vmatrix} = 0 \tag{31}$$

where $\bar{c}_{pq} = c_{pq}/c_{66}$.

The dispersion curves of the essentially symmetric family of modes are associated with the dispersion relation (30), while the equation (31) yields the dispersion curves of the essentially anti-symmetric family of modes.

The cut off frequency, for $z = 0$, are the exact ones. In order to have the correct slope of the lowest flexural branch at $z = 0$ one must set

$$k_1 = \pi/4. \tag{32}$$

For $\Omega \gg 1$ and $z \gg 1$, the phase velocity of the lowest flexural branch approaches to [6], from (31),

$$k_2 = \Omega/z = c^2\rho/c_{66} = 0{\cdot}901. \tag{33}$$

Hence the phase velocity of the lowest flexural branch approaches that of surface wave in monoclinic crystals.

The dispersion curves are then computed for AT–cut quartz crystal plate and are compared with the dispersion curves obtained from the three-dimensional exact theory in Fig. 1 for both antisymmetric and symmetric families of modes. It can be seen that the two sets of curves match quite well for $\Omega < 1{\cdot}5$.

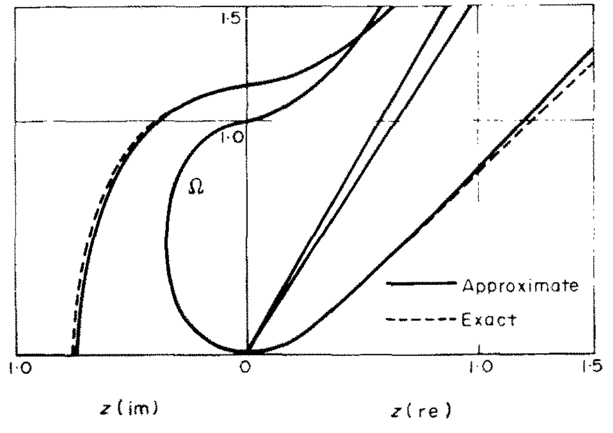

Fig. 1. Dispersion curves of the essentially symmetric and antisymmetric families of modes for the first order approximate theory.

Second order theory

Following the truncation process described above, we set $N = 2$ in (18). Then the energy densities for the second order theory are

$$\begin{aligned}\bar{U}^{(2)} &= \tfrac{1}{2}[T_{ij}^{(0)}S_{ij}^{(0)} + T_{ij}^{(1)}S_{ij}^{(1)} + T_{ij}^{(2)}S_{ij}^{(2)} + \bar{T}_{ij}^{(1)}\bar{S}_{ij}^{(1)} + \bar{T}_{ij}^{(2)}\bar{S}_{ij}^{(2)}] \\ \bar{K}^{(2)} &= \tfrac{1}{2}\rho[k_2 - u_{i,t}^{(0)}u_{i,t}^{(0)} + u_{i,t}^{(1)}u_{i,t}^{(1)} + u_{i,t}^{(2)}u_{i,t}^{(2)}]\end{aligned} \tag{34}$$

and the stress equations of motion become

$$T_{ij,i}^{(0)} + \frac{1}{b} F_j^{(0)} = \rho k_2^{-p} u_{j,tt}^{(0)}$$
$$T_{ij,i}^{(n)} - \frac{n\pi}{2b} \overline{T}_{2j}^{(n)} + \frac{1}{b} F_j^{(n)} = \rho u_{j,tt}^{(n)} \qquad n = 1, 2. \tag{35}$$

The strain–displacement and stress–strain relations are obtained from (9) and (11), respectively, with strain and displacement components of order higher than two discarded. The displacement equations of motion are then obtained similarly as in the first order theory.

$$\tfrac{1}{2}c_{ijkl}\left[u_{k,l}^{(0)} + u_{l,k}^{(0)} + \frac{2}{b} k_1(\delta_{2l}u_k^{(1)} + \delta_{2k}u_l^{(1)})\right]_{,i} + \frac{1}{b} F_j^{(0)} = \rho k_2^{-p} u_{j,tt}^{(0)}$$

$$\tfrac{1}{2}c_{ijkl}\left[k_2^{p}(u_{k,l}^{(1)} + u_{l,k}^{(1)}) + \frac{8}{3b}(\delta_{2l}u_k^{(2)} + \delta_{2k}u_l^{(2)})\right]_{,i}$$
$$-\frac{\pi}{4b} c_{2jkl}\left[\frac{\pi}{2b}(\delta_2 l u_k^{(1)} + \delta_{2k}u_l^{(1)}) + \frac{4}{\pi} k_1 (u_{k,l}^{(0)} + u_{l,k}^{(0)}) - \frac{4}{3\pi}(u_{k,l}^{(2)} + u_{l,k}^{(2)})\right] + \frac{1}{b} F_j^{(1)} = \rho u_{j,tt}^{(1)}$$

$$\tfrac{1}{2}c_{ijkl}\left[u_{k,l}^{(2)} + u_{l,k}^{(2)} - \frac{2}{3b}(\delta_{2l}u_k^{(1)} + \delta_{2k}u_l^{(1)})\right]_{,i}$$
$$-\frac{\pi}{2b} c_{2jkl}\left[\frac{\pi}{b}(\delta_{2l}u_k^{(2)} + \delta_{2k}u_l^{(2)}) + \frac{8}{3\pi}(u_{k,l}^{(1)} + u_{l,k}^{(1)})\right] + \frac{1}{b} F_j^{(2)} = \rho u_{j,tt}^{(2)}. \tag{36}$$

Consider the straight crested waves propagating in the x_1 direction by setting

$$\begin{aligned} u_j^{(0)} &= A_j^{(0)} e^{i(\xi x_1 - \omega t)} \\ u_j^{(1)} &= -iA_j^{(1)} e^{i(\xi x_1 - \omega t)} \\ u_j^{(2)} &= A_j^{(2)} e^{i(\xi x_1 - \omega t)} \\ F_j^{(0)} &= F_j^{(1)} = F_j^{(2)} = 0. \end{aligned} \tag{37}$$

The displacement equations of motion will separate into two uncoupled sets and the dispersion relations of the essentially symmetric family, equation (38), and the antisymmetric family of modes are obtained.

$$\begin{vmatrix} \bar{c}_{11}z^2 - \Omega^2 & -\bar{c}_{12}\frac{4}{\pi}k_1 z & -\bar{c}_{14}\frac{4}{\pi}k_1 z & 0 \\ -\bar{c}_{12}\frac{4}{\pi}k_1 z & k_2 z^2 + \bar{c}_{22} - \Omega^2 & \bar{c}_{56}z^2 + \bar{c}_{24} & \frac{4}{3\pi}(\bar{c}_{12}+4)z \\ -\bar{c}_{14}\frac{4}{\pi}k_1 z & \bar{c}_{56}z^2 + \bar{c}_{24} & \bar{c}_{55}z^2 + \bar{c}_{44} - \Omega^2 & \frac{4}{3\pi}(\bar{c}_{14}+4\bar{c}_{56})z \\ 0 & \frac{4}{3\pi}(\bar{c}_{12}+4)z & \frac{4}{3\pi}(\bar{c}_{14}+4\bar{c}_{56})z & \bar{c}_{11}z^2 + 4 - \Omega^2 \end{vmatrix} = 0. \tag{38}$$

The dispersion curves yield the exact cut-off frequencies at $z = 0$.

As both $z \ll 1$ and $\Omega \ll 1$, (38) gives the exact slopes of the branches, if $k_1 = \pi/4$. As $z \gg 1$ and $\Omega \gg 1$, it is found from (38) that the phase velocity of the lowest extensional branch approaches to

$$\Omega/z = k_2 = 0{\cdot}901$$

which is the phase velocity of the surface wave in monoclinic crystals[2].

The dispersion curves of the essentially symmetric family of modes are presented and compared with the results from the three-dimensional theory of elasticity in Fig. 2. The imaginary dispersion curves are not fully developed for this order theory. Only one complex branch is found.

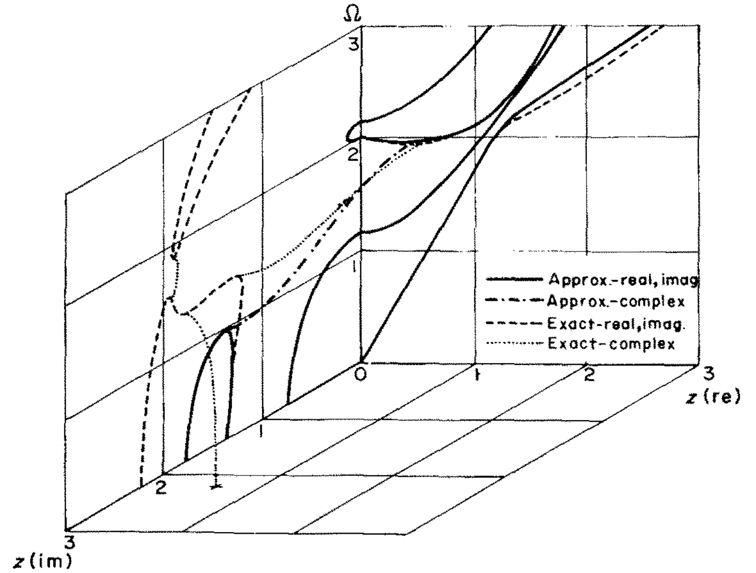

Fig. 2. Dispersion curves of the essentially symmetric family of modes for the second order approximate theory.

It may be noted that the same value of $k_1 = \pi/4$ is used for correcting the slopes of both the lowest flexural and the lowest extensional branches at $z = 0$ and $\Omega = 0$, and the same value of k_2 is used for correcting the phase velocities of the same two branches when the values of z and Ω become large. As it shall be seen, no additional corrections are needed in the higher branches of the higher order theories.

Third order theory

The energy densities are defined as

$$\bar{U}^{(3)} = \tfrac{1}{2}[T_{ij}^{(0)}S_{ij}^{(0)} + T_{ij}^{(1)}S_{ij}^{(1)} + T_{ij}^{(2)}S_{ij}^{(2)} + T_{ij}^{(3)}S_{ij}^{(3)} + \bar{T}_{ij}^{(1)}\bar{S}_{ij}^{(1)} + \bar{T}_{ij}^{(2)}\bar{S}_{ij}^{(2)} + \bar{T}_{ij}^{(3)}\bar{S}_{ij}^{(3)}] \quad (39)$$

$$\bar{K}^{(3)} = \tfrac{1}{2}\rho[k_2^{-p}u_{i,t}^{(0)}u_{i,t}^{(0)} + u_{i,t}^{(1)}u_{i,t}^{(1)} + u_{i,t}^{(2)}u_{i,t}^{(2)} + u_{i,t}^{(3)}u_{i,t}^{(3)}]$$

and the stress equation of motion are:

$$
\begin{aligned}
&T_{ij,i}^{(0)} + \frac{1}{b} F_j^{(0)} = \rho k_2^{-p} u_{j,tt}^{(0)} \\
&T_{ij,i}^{(n)} - \frac{n\pi}{2b} \bar{T}_{2j}^{(n)} + \frac{1}{b} F_j^{(n)} = \rho u_{j,tt}^{(n)} \qquad n = 1, 2, 3
\end{aligned}
\tag{40}
$$

Discarding the components of strain and displacement of order higher than three, the stress–strain and stress–displacement relations are then obtained. The displacement equations of motion are then:

$$
\begin{aligned}
&\tfrac{1}{2}c_{ijkl}\left[u_{k,l}^{(0)} + u_{l,k}^{(0)} + \frac{2}{b} k_1(\delta_{2l}u_k^{(1)} + \delta_{2k}u_l^{(1)}) + \frac{2}{b}(\delta_{2l}u_k^{(3)} + \delta_{2k}u_l^{(3)})\right]_{,i} \\
&\qquad + \frac{1}{b} F_j^{(0)} = \rho k_2^{-p} u_{j,tt}^{(0)} \\
&\tfrac{1}{2}c_{ijkl}\left[k_2^p(u_{k,l}^{(1)} + u_{l,k}^{(1)}) + \frac{8}{3b}(\delta_{2l}u_k^{(2)} + \delta_{2k}u_l^{(2)})\right]_{,i} \\
&\quad - \frac{\pi}{4b} c_{2jkl}\left[\frac{\pi}{2b}(\delta_{2l}u_k^{(1)} + \delta_{2k}u_l^{(1)}) + \frac{4}{\pi} k_1(u_{k,l}^{(0)} + u_{l,k}^{(0)}) - \frac{4}{3\pi}(u_{k,l}^{(2)} + u_{l,k}^{(2)})\right] \\
&\qquad + \frac{1}{b} F_j^{(1)} = \rho u_{j,tt}^{(1)} \\
&\tfrac{1}{2}c_{ijkl}\left[u_{k,l}^{(2)} + u_{l,k}^{(2)} - \frac{2}{3b}(\delta_{2l}u_k^{(1)} + \delta_{2k}u_l^{(1)}) + \frac{18}{5b}(\delta_{2l}u_k^{(3)} + \delta_{2k}u_l^{(3)})\right]_{,i} \\
&\quad - \frac{\pi}{2b} c_{2jkl}\left[\frac{\pi}{b}(\delta_{2l}u_k^{(2)} + \delta_{2k}u_l^{(2)}) + \frac{8}{3\pi}(u_{k,l}^{(1)} + u_{l,k}^{(1)}) - \frac{8}{5\pi}(u_{k,l}^{(3)} + u_{l,k}^{(3)})\right] + \frac{1}{b} F_j^{(2)} = \rho u_{j,tt}^{(2)} \\
&\tfrac{1}{2}c_{ijkl}\left[u_{k,l}^{(3)} + u_{l,k}^{(3)} - \frac{8}{5b}(\delta_{2l}u_k^{(2)} + \delta_{2k}u_l^{(2)})\right]_{,i} \\
&\quad - \frac{3\pi}{4b} c_{2jkl}\left[\frac{3\pi}{2b}(\delta_{2l}u_k^{(3)} + \delta_{2k}u_l^{(3)}) + \frac{4}{3\pi}(u_{k,l}^{(0)} + u_{l,k}^{(0)}) + \frac{12}{5\pi}(u_{k,l}^{(2)} + u_{l,k}^{(2)})\right] + \frac{1}{b} F_j^{(3)} = \rho u_{j,tt}^{(3)}.
\end{aligned}
\tag{41}
$$

The propagation of straight crested waves in the x_1 direction only is obtained by setting:

$$
\begin{aligned}
&u_j^{(n)} = A_j^{(n)} e^{i(\xi x_1 - \omega t)} && n = 0, 2 \\
&u_j^{(n)} = -\mathrm{i} A_j^{(n)} e^{i(\xi x_1 - \omega t)} && n = 1, 3 \\
&F_j^{(n)} = 0 && n = 0, 1, 2, 3.
\end{aligned}
\tag{42}
$$

The dispersion relation for the essentially antisymmetric family of modes is then, in dimensionless coordinates, presented in (43).

$$
\begin{vmatrix}
z^2 - \frac{1}{k_2}\Omega^2 & \bar{c}_{56} z^2 & -\frac{4}{\pi} k_1 z & 0 & 0 & -\frac{4}{\pi} z \\
\bar{c}_{56} z^2 & \bar{c}_{55} z^2 - \Omega^2 & -\bar{c}_{56} \frac{4}{\pi} k_1 z & 0 & 0 & -\bar{c}_{56} \frac{4}{\pi} z \\
-\frac{4}{\pi} k_1 z & -\bar{c}_{56} \frac{4}{\pi} k_1 z & \bar{c}_{11} z^2 + 1 - \Omega^2 & \frac{4}{3\pi}(4\bar{c}_{12} + 1)z & \frac{4}{3\pi}(4\bar{c}_{14} + \bar{c}_{56})z & 0 \\
0 & 0 & \frac{4}{3\pi}(4\bar{c}_{12} + 1)z & z^2 + 4\bar{c}_{22} - \Omega^2 & \bar{c}_{56} z^2 + 4\bar{c}_{24} & -\frac{4}{5\pi}(4\bar{c}_{12} + 9)z \\
0 & 0 & \frac{4}{3\pi}(4\bar{c}_{14} + \bar{c}_{56})z & \bar{c}_{56} z^2 + 4\bar{c}_{24} & \bar{c}_{55} z^2 + 4\bar{c}_{44} - \Omega^2 & -\frac{4}{5\pi}(4\bar{c}_{14} + 9\bar{c}_{56})z \\
-\frac{4}{\pi} z & -\bar{c}_{56} \frac{4}{\pi} z & 0 & -\frac{4}{5\pi}(4\bar{c}_{12} + 9)z & -\frac{4}{5\pi}(4\bar{c}_{14} + 9\bar{c}_{56})z & \bar{c}_{11} z^2 + 9 - \Omega^2
\end{vmatrix} = 0 \qquad (43)
$$

The dispersion curves for the essentially antisymmetric family of modes for the AT–cut quartz crystal plate are computed from (43) and are compared with those computed from the three-dimensional theory. There is no complex branch of the dispersion curves found for third order approximation. The two sets of curves match very well as it is seen from Fig. 3.

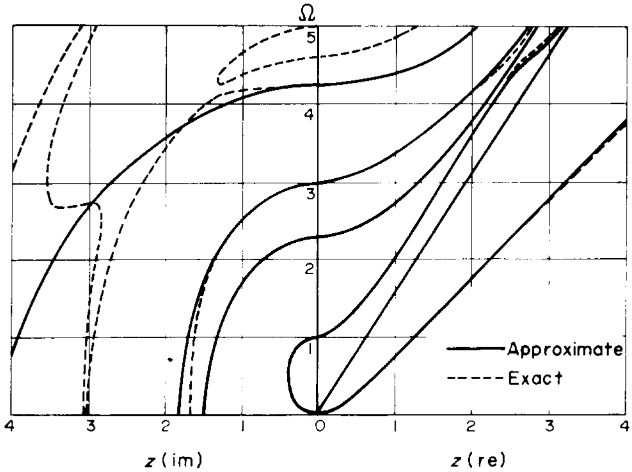

Fig. 3. Dispersion curves of the essentially antisymmetric family of modes for the third order approximate theory.

Fourth order theory

The energy densities are

$$\bar{U}^{(4)} = \tfrac{1}{2}[T_{ij}^{(0)}S_{ij}^{(0)} + T_{ij}^{(1)}S_{ij}^{(1)} + T_{ij}^{(2)}S_{ij}^{(2)} + T_{ij}^{(3)}S_{ij}^{(3)} + T_{ij}^{(4)}S_{ij}^{(4)} + \bar{T}_{ij}^{(1)}\bar{S}_{ij}^{(1)} + \bar{T}_{ij}^{(2)}\bar{S}_{ij}^{(2)} + \bar{T}_{ij}^{(3)}\bar{S}_{ij}^{(3)} + \bar{T}_{ij}^{(4)}\bar{S}_{ij}^{(4)}] \quad (44)$$

$$\bar{K}^{(4)} = \tfrac{1}{2}\rho[k_2^{-p}u_{i,t}^{(0)}u_{i,t}^{(0)} + u_{i,t}^{(1)}u_{i,t}^{(1)} + u_{i,t}^{(2)}u_{i,t}^{(2)} + u_{i,t}^{(3)}u_{i,t}^{(3)} + u_{i,t}^{(4)}u_{i,t}^{(4)}]$$

and the stress equations of motion are:

$$\begin{aligned} &T_{ij,i}^{(0)} + \frac{1}{b}F_j^{(0)} = \rho k_2^{-p}u_{j,tt}^{(0)} \\ &T_{ij,i}^{(n)} - \frac{n\pi}{2b}\bar{T}_{2j}^{(n)} + \frac{1}{b}F_j^{(n)} = \rho u_{j,tt}^{(n)} \qquad n = 1, 2, 3, 4. \end{aligned} \quad (45)$$

The displacement equations of motion are then obtained by substituting the stress–displacement relations, where components of order higher than four are discarded, into the stress equations of motion.

$$\tfrac{1}{2}c_{ijkl}\left[u_{k,l}^{(0)}+u_{l,k}^{(0)}+\frac{2}{b}k_1(\delta_{2l}u_k^{(1)}+\delta_{2k}u_l^{(1)})+\frac{2}{b}(\delta_{2l}u_k^{(3)}+\delta_{2k}u_l^{(3)})\right]_{,i}+\frac{1}{b}F_j^{(0)}=\rho k_2^{-p}u_{j,tt}^{(0)}$$

$$\tfrac{1}{2}c_{ijkl}\left[k_2^{p}(u_{k,l}^{(1)}+u_{l,k}^{(1)})+\frac{8}{3b}(\delta_{2l}u_k^{(2)}+\delta_{2k}u_l^{(2)})+\frac{32}{15b}(\delta_{2l}u_k^{(4)}+\delta_{2k}u_l^{(4)})\right]_{,i}$$

$$-\frac{\pi}{4b}c_{2jkl}\left[\frac{\pi}{2b}(\delta_{2l}u_k^{(1)}+\delta_{2k}u_l^{(1)})+\frac{4}{\pi}k_1(u_{k,l}^{(0)}+u_{l,k}^{(0)})-\frac{4}{3\pi}(u_{k,l}^{(2)}+u_{l,k}^{(2)})\right.$$

$$\left.-\frac{4}{15\pi}(u_{k,l}^{(4)}+u_{l,k}^{(4)})\right]+\frac{1}{b}F_j^{(1)}=\rho u_{j,tt}^{(1)}$$

$$\tfrac{1}{2}c_{ijkl}\left[u_{k,l}^{(2)}+u_{l,k}^{(2)}-\frac{2}{3b}(\delta_{2l}u_k^{(1)}+\delta_{2k}u_l^{(1)})+\frac{18}{5b}(\delta_{2l}u_k^{(3)}+\delta_{2k}u_l^{(3)})\right]_{,i}$$

$$-\frac{\pi}{2b}c_{2jkl}\left[\frac{\pi}{b}(\delta_{2l}u_k^{(2)}+\delta_{2k}u_l^{(2)})+\frac{8}{3\pi}(u_{k,l}^{(1)}+u_{l,k}^{(1)})-\frac{8}{5\pi}(u_{k,l}^{(3)}+u_{l,k}^{(3)})\right]+\frac{1}{b}F_j^{(2)}=\rho u_{j,tt}^{(2)}$$

$$\tfrac{1}{2}c_{ijkl}\left[u_{k,l}^{(3)}+u_{l,k}^{(3)}-\frac{8}{5b}(\delta_{2l}u_k^{(2)}+\delta_{2k}u_l^{(2)})+\frac{32}{7b}(\delta_{2l}u_k^{(4)}+\delta_{2k}u_l^{(4)})\right]_{,i}$$

$$-\frac{3\pi}{4b}c_{2jkl}\left[\frac{3\pi}{2b}(\delta_{2l}u_k^{(3)}+\delta_{2k}u_l^{(3)})+\frac{4}{3\pi}(u_{k,l}^{(0)}+u_{l,k}^{(0)})+\frac{12}{5\pi}(u_{k,l}^{(2)}+u_{l,k}^{(2)})\right. \tag{46}$$

$$\left.-\frac{12}{7\pi}(u_{k,l}^{(4)}+u_{l,k}^{(4)})\right]+\frac{1}{b}F_j^{(3)}=\rho u_{j,tt}^{(3)}$$

$$\tfrac{1}{2}c_{ijkl}\left[u_{k,l}^{(4)}+u_{l,k}^{(4)}-\frac{2}{15b}(\delta_{2l}u_k^{(1)}+\delta_{2k}u_l^{(1)})-\frac{18}{7b}(\delta_{2l}u_k^{(3)}+\delta_{2k}u_l^{(3)})\right]_{,i}$$

$$-\frac{\pi}{b}c_{2jkl}\left[\frac{2\pi}{b}(\delta_{2l}u_k^{(4)}+\delta_{2k}u_l^{(4)})+\frac{16}{5\pi}(u_{k,l}^{(1)}+u_{l,k}^{(1)})+\frac{16}{7\pi}(u_{k,l}^{(3)}+u_{l,k}^{(3)})\right]+\frac{1}{b}F_j^{(4)}=\rho u_{j,tt}^{(4)}$$

By setting

$$\begin{aligned} u_j^{(n)} &= A_j^{(n)}e^{i(\xi x_1-\omega t)} && n=0,2,4\\ u_j^{(n)} &= -iA_j^{(n)}e^{i(\xi x_1-\omega t)} && n=1,3\\ F_j^{(n)} &= 0 && n=0,1,2,3,4 \end{aligned} \tag{47}$$

$$\begin{vmatrix}
\bar{c}_{11}z^2-\Omega^2 & -\bar{c}_{12}\dfrac{4}{\pi}k_1z & -\bar{c}_{14}\dfrac{4}{\pi}k_1z & 0 & -\bar{c}_{12}\dfrac{4}{\pi}z & -\bar{c}_{14}\dfrac{4}{\pi}z & 0 \\
-\bar{c}_{12}\dfrac{4}{\pi}k_1z & k_2z^2+\bar{c}_{22}-\Omega^2 & \bar{c}_{56}z^2+\bar{c}_{24} & \dfrac{4}{3\pi}(\bar{c}_{12}+4)z & 0 & 0 & \dfrac{4}{15\pi}(\bar{c}_{12}+16)z \\
-\bar{c}_{14}\dfrac{4}{\pi}k_1z & \bar{c}_{56}z^2+\bar{c}_{24} & \bar{c}_{55}z^2+\bar{c}_{44}-\Omega^2 & \dfrac{4}{3\pi}(\bar{c}_{14}+4\bar{c}_{56})z & 0 & 0 & \dfrac{4}{15\pi}(\bar{c}_{14}+16\bar{c}_{56})z \\
0 & \dfrac{4}{3\pi}(\bar{c}_{12}+4)z & \dfrac{4}{3\pi}(\bar{c}_{14}+4\bar{c}_{56})z & \bar{c}_{11}z^2+4-\Omega^2 & -\dfrac{4}{5\pi}(9\bar{c}_{12}+4)z & -\dfrac{4}{5\pi}(9\bar{c}_{14}+4\bar{c}_{56})z & 0 \\
-\bar{c}_{12}\dfrac{4}{\pi}z & 0 & 0 & -\dfrac{4}{5\pi}(9\bar{c}_{12}+4)z & z^2+9\bar{c}_{22}-\Omega^2 & \bar{c}_{55}z^2+9\bar{c}_{44} & \dfrac{4}{7\pi}(9\bar{c}_{12}+16)z \\
-\bar{c}_{14}\dfrac{4}{\pi}z & 0 & 0 & -\dfrac{4}{5\pi}(9\bar{c}_{14}+4\bar{c}_{56})z & \bar{c}_{56}z^2+9\bar{c}_{24} & \bar{c}_{55}z^2+9\bar{c}_{44}-\Omega^2 & \dfrac{4}{7\pi}(9\bar{c}_{14}+16\bar{c}_{56})z \\
0 & \dfrac{4}{15\pi}(\bar{c}_{12}+16)z & \dfrac{4}{15\pi}(\bar{c}_{14}+16\bar{c}_{56})z & 0 & \dfrac{4}{7\pi}(9\bar{c}_{12}+16)z & \dfrac{4}{7\pi}(9\bar{c}_{14}+16\bar{c}_{56})z & \bar{c}_{11}z^2+16-\Omega^2
\end{vmatrix}=0 \tag{48}$$

we obtain two sets of dispersion relations for the propagation of the straight crested waves in the monoclinic crystal plate. The dispersion relation for the essentially symmetric family of modes is presented in (48).

The dispersion curves are computed from (48) for the AT–cut quartz plate and are compared with the exact ones. It can be observed that the lower branches match with the "exact" curves better than the corresponding branches in the second order theory, since the higher order theory not only accommodates additional branches at higher frequencies but also improves the accuracy of the lower branches.

The discrepancy for the complex branches are mostly due to the difference in the real parts. The matching of the imaginary parts is very close. The mode shape corresponding to a complex wave number can be expressed as the product of trigonometric and hyperbolic functions. Hence the mode has the shape of a oscillatory variation within the envelope of the hyperbolic function, and therefore in the case of matching the complex branches it is more important to have close agreement of the imaginary parts rather than the real ones, since the former control the amplitudes of the modes.

6. GENERATION OF HIGHER ORDER DISPERSION RELATIONS

An approximate theory of the order N (any $N > 0$) can be generated in a systematic manner by using the truncation process (18). No additional correction coefficients are needed for any higher order theory. The dispersion relations for the propagation of straight crested waves in the x_1 direction only in crystal plates with monoclinic symmetry are then obtained by substitution of the wave form solutions into the displacement equations of motion. Since the process becomes tedius when N is large, a general method for generating the dispersion relations of monoclinic crystal plates is described below.

The matrices of the dispersion relations are symmetric and their elements may be classified into four groups: A_{mn}, B_{mn} for elements on the main diagonal (the later is a three-term group, two diagonal terms and one off-diagonal term); C_{mn} (two terms in one row) and D_{mn} (two terms in one column) for the elements off main diagonal.

In terms of these groups the dispersion relations of an N-th order theory for the essentially symmetric and the essentially antisymmetric families of modes, respectively, can be written as:

$$
\begin{array}{cc|cccccccc|}
 & & & & & & m & & & \\
 & & 0 & 1 & 2 & 3 & 4 & 5 & \cdots & N \\
 & 0 & A_{00} & C_{01} & 0 & C_{03} & 0 & C_{05} & \cdots & 0 \\
 & 1 & & B_{11} & D_{12} & 0 & D_{14} & 0 & \cdots & D_{1N} \\
 & 2 & & & A_{22} & C_{23} & 0 & C_{25} & \cdots & 0 \\
 & 3 & & & & B_{33} & D_{34} & 0 & \cdots & D_{3N} \\
n & 4 & & & & & A_{44} & C_{45} & \cdots & 0 \\
 & 5 & & & & & & B_{55} & \cdots & D_{5N} \\
 & \vdots & & & & & & & \cdots & \\
 & N & & & & & & & & A_{NN}
\end{array} = 0 \qquad (49)
$$

and

$$
\begin{array}{cc|cccccccc|}
 & & & & & m & & & & \\
 & & 0 & 1 & 2 & 3 & 4 & 5 & \cdots & N \\
 & 0 & B_{00} & D_{01} & 0 & D_{03} & 0 & D_{05} & \cdots & D_{0N} \\
 & 1 & & A_{11} & C_{12} & 0 & C_{14} & 0 & \cdots & 0 \\
 & 2 & & & B_{22} & D_{23} & 0 & D_{25} & \cdots & D_{2N} \\
 & 3 & & & & A_{33} & C_{34} & 0 & \cdots & 0 \\
n & 4 & & & & & B_{44} & D_{45} & \cdots & D_{4N} \\
 & 5 & & & & & & A_{55} & \cdots & 0 \\
 & \vdots & & & & & & & \cdots & \\
 & N & & & & & & & & A_{NN}
\end{array} = 0. \qquad (50)
$$

The terms in the groups are defined as follows ($0 \leqslant m, n \leqslant N$, for any $N > 0$):

$$A_{nm} = \bar{c}_{11}z^2 + n^2 - \Omega^2 \qquad (n = m)$$

$$B_{nm} = \begin{Bmatrix} z^2 + n^2\bar{c}_{22} - \Omega^2 & \bar{c}_{56}z^2 + n^2\bar{c}_{24} \\ & \bar{c}_{55}z^2 + n^2\bar{c}_{44} - \Omega^2 \end{Bmatrix} \qquad (n = m)$$

$$C_{nm} = \left\{ \mp \frac{4}{(m^2 - n^2)\pi}(m^2\bar{c}_{12} + n^2)z \qquad \mp \frac{4}{(m^2 - n^2)\pi}(m^2\bar{c}_{14} + n^2\bar{c}_{56})z \right\} \qquad (n \neq m) \quad (n + m = \text{odd})$$

$$D_{nm} = \begin{Bmatrix} \pm \dfrac{4}{(m^2 - n^2)\pi}(n^2\bar{c}_{12} + m^2)z \\ \pm \dfrac{4}{(m^2 - n^2)\pi}(n^2\bar{c}_{14} + m^2\bar{c}_{56})z \end{Bmatrix} \qquad (n \neq m) \quad (n + m = \text{odd}).$$

The upper sign in the last two groups is applied to the essentially symmetric family of modes and the lower sign is applied to the essentially antisymmetric family of modes. The elements of determinants of dispersion relations are then generated with the help of (49) and (50). It shall be noted that the correction coefficients k_1 and k_2 are applied in terms B_{00}, B_{11}, C_{01}, D_{01} as is seen, for example, in dispersion relations (43) and (48). The dispersion relation is in the form of a determinant, whose order s, for N-th order theory, is given by

$$s = \frac{1}{2}\left[3(N + 1) \pm \cos^2\frac{N\pi}{2}\right].$$

The dispersion relations for the fifth, sixth, ninth and tenth order theories are generated by this method. The dispersion curves for the essentially antisymmetric (fifth and ninth order theories) and essentially symmetric (sixth and tenth order theories) families of modes are computed for AT–cut quartz crystal plate and compared with the curves obtained from the three-dimensional theory (Figs. 5–8). The two sets of curves match very well

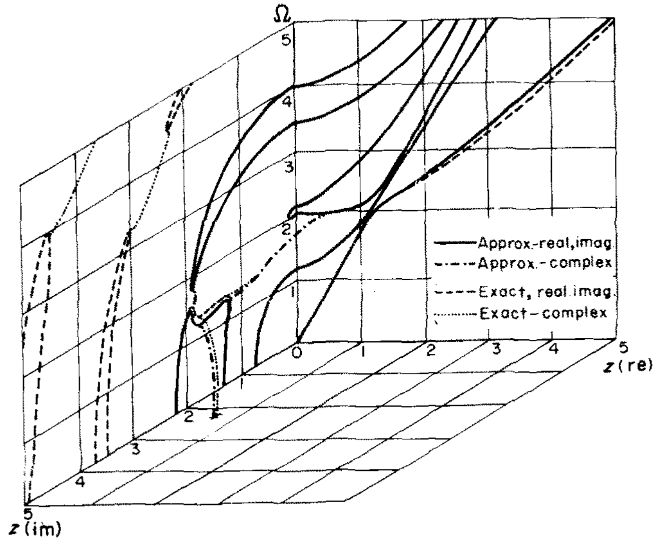

Fig. 4. Dispersion curves of the essentially symmetric family of modes for the fourth order approximate theory.

except for the complex branch in the fifth order dispersion curves for the essentially antisymmetric family of modes, since the imaginary part of this branch is not yet fully developed; it develops in the ninth order theory (Fig. 7).

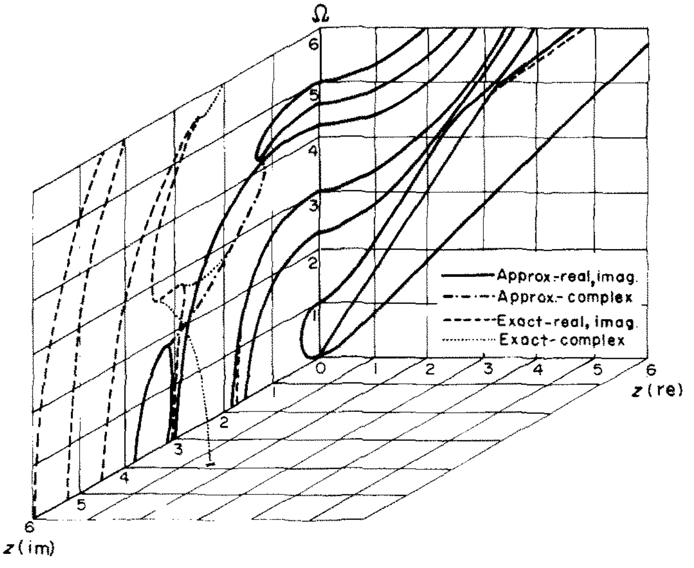

Fig. 5. Dispersion curves of the essentially antisymmetric family of modes for the fifth order approximate theory.

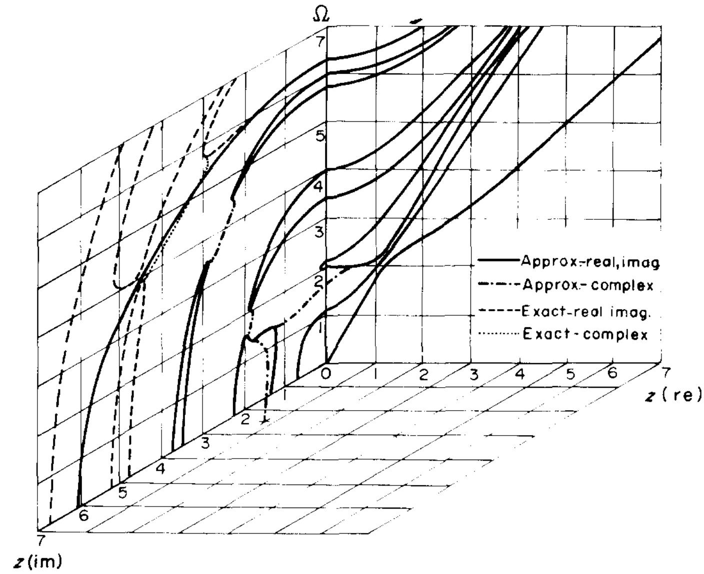

Fig. 6. Dispersion curves of the essentially symmetric family of modes for the sixth order approximate theory.

Due to the increasing computational difficulties only the real and imaginary parts of the dispersion curves are shown in Figs. 7 and 8, for the essentially antisymmetric family of modes for the ninth order theory and the essentially symmetric family of modes for the tenth order theory, respectively.

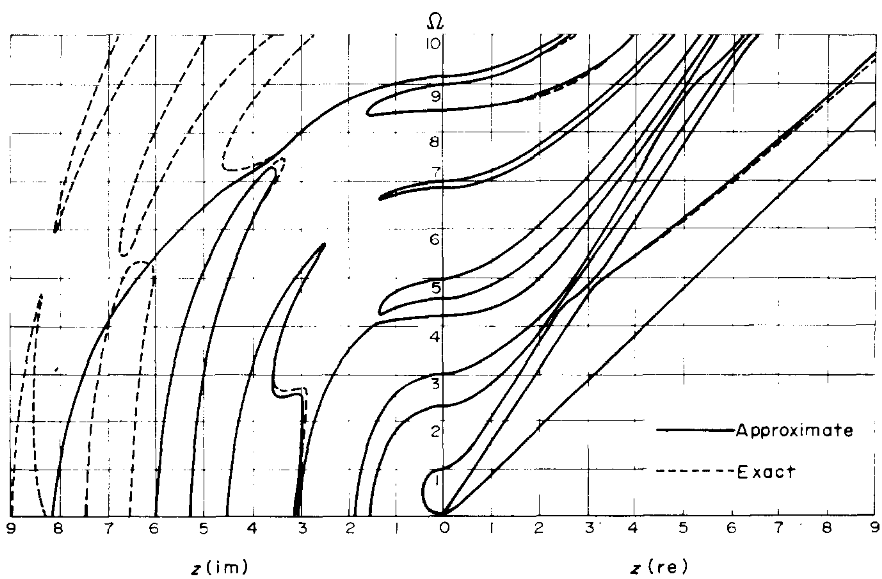

Fig. 7. Dispersion curves of the essentially antisymmetric family of modes for the ninth order approximate theory.

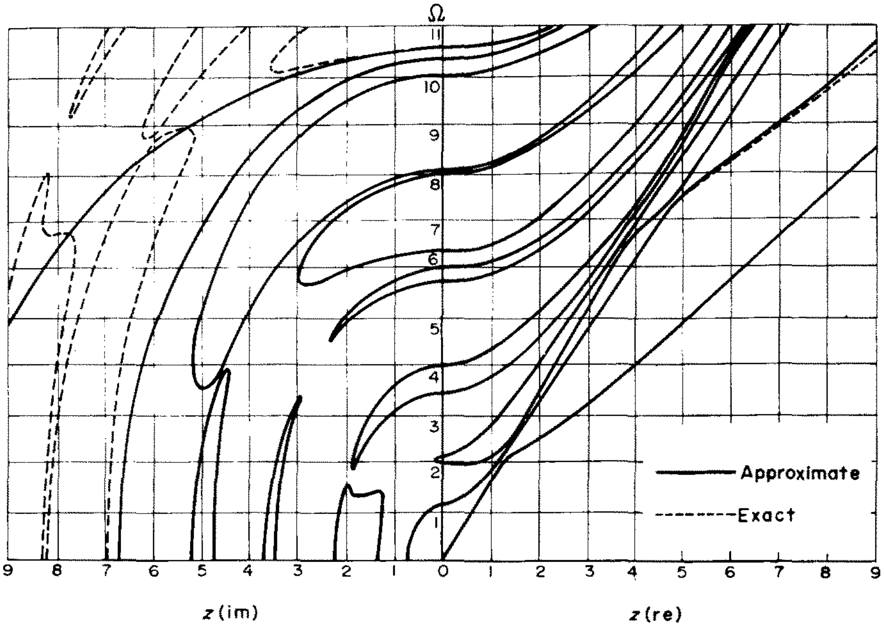

Fig. 8. Dispersion curves of the essentially symmetric family of modes for the tenth order approximate theory.

7. CONCLUSIONS

The expansion of components of displacement in a series of simple thickness modes enabled us to obtain an infinite set of stress equations of motion, strain–displacement relations, constitutive relations and displacement equations of motion. The dispersion relations for crystal plates with monoclinic symmetry were obtained for approximate theories of order one up to the order four.

The dispersion curves for AT–cut quartz plate were then computed and compared with the curves obtained from the exact three-dimensional theory. The close agreement of both sets of curves indicates that the applicable range of frequencies for an Nth order theory can be set at $\Omega \leqslant N+\frac{1}{2}$.

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Абстракт — Применяется разложение смещения в ряды, выраженные через элементарные колебания по толщине, с целью получения приближенных, двухмерных уравнений движения для кристаллических пластинок исходя из трехмерной теории упругости. Даются приближенные теории от первого до четвертого порядка. Исследуются кривы дисперсии для кварцевой пластинки AT среза и сравниваются с решениями трехмерных уравнений для бесконечной пластинки.